TRANSMISSION EIGENVALUES FOR OPERATORS WITH CONSTANT COEFFICIENTS

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ABSTRACT. In this paper we study the interior transmission problem and transmission eigenvalues for multiplicative perturbations of linear partial differential operator of order ≥ 2 with constant real coefficients. Under suitable growth conditions on the symbol of the operator and the perturbation, we show the discreteness of the set of transmission eigenvalues and derive sufficient conditions on the existence of transmission eigenvalues. We apply these techniques to the case of the biharmonic operator and the Dirac system. In the hypoelliptic case we present a connection to scattering theory.

1. INTRODUCTION

Let $V \in L^\infty(\mathbb{R}^n)$ be compactly supported with supp $(V) = \overline{\Omega}$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain, and let $P_0(D)$ be a partial differential operator of order $m \geq 2$ with constant real coefficients

$$P_0(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha, \quad a_\alpha \in \mathbb{R}, \quad D_j = -i\frac{\partial}{\partial x_j}, \quad j = 1, \ldots, n.$$

For any $\varphi \in C_0^\infty(\Omega)$, we set

$$\|\varphi\|_{P_0} = \|P_0(D)\varphi\|. \tag{1.1}$$

Here and in what follows the notation $\|\cdot\|$ stands for the standard L^2-norm. The completion of $C_0^\infty(\Omega)$ with respect to the norm (1.1) is denoted by $H_0^{P_0}(\Omega)$.

The *interior transmission problem* is the following degenerate boundary value problem,

$$\begin{aligned}
(P_0 - \lambda)v &= 0 \quad \text{in} \quad \Omega, \\
(P_0 + V - \lambda)w &= 0 \quad \text{in} \quad \Omega, \\
v - w &\in H_0^{P_0}(\Omega).
\end{aligned} \tag{1.2}$$

We say that $\lambda \in \mathbb{C}$ is a *transmission eigenvalue* if the problem (1.2) has non-trivial solutions $0 \neq v \in L^2_{\text{loc}}$ and $0 \neq w \in L^2_{\text{loc}}$.

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The problem (1.2) arises naturally in the study of inverse scattering theory for the operator $P_0 + V$, when $P_0 = -\Delta$. In this context, the significance of the notion of a transmission eigenvalue is twofold. On the one hand, in reconstruction algorithms of inverse scattering theory, such as the linear sampling method of Colton and Kirsch [2, 8], and the factorization method of Kirsch [20], transmission eigenvalues correspond to frequencies that one needs to avoid. The fact that they form a discrete set makes this possible. On the other hand, the knowledge of the transmission eigenvalues carries information about the scatterer [23], and in the radially symmetric case, they determine the potential completely [4], see also [21] for the previous work in this direction. It is therefore natural to investigate the question of existence of transmission eigenvalues, and relate their properties to the properties of the scatterer.

The interior transmission problem was first introduced in 1988 by Colton and Monk [10] in connection with an inverse scattering problem for the reduced wave equation. They were led to this problem when studying the injectivity of the far field operator. The fact that the interior transmission eigenvalues form a discrete set was shown in the Helmholz case in [9]. The problem of existence of transmission eigenvalues, however, remained unsolved until recently. When $P_0 = -\Delta$, Päivärinta and Sylvester in [23] proved the first existence result, and soon thereafter, the existence of an infinite set of real transmission eigenvalues was established by Cakoni, Gintides, and Haddar in [5]. We would also like to mention recent results on transmission eigenvalues for Maxwell's equations, as well as for the Helmholtz equation, in the presence of cavities [3, 6, 19].

In this paper we consider the case of quite general operators P_0 with constant coefficients, for which a well developed scattering theory is available [15]. In particular, we can allow non-elliptic operators as well as operators of higher order. We then show the discreteness of the set of transmission eigenvalues, and derive sufficient conditions for the existence of an infinite set of real transmission eigenvalues. The extension to the case of general operators with constant coefficients seems natural, as in applications one frequently encounters such operators which could be either non-elliptic, or elliptic of higher order. We also illustrate the techniques developed by applying them in two cases of physically significant operators, namely the biharmonic operator and the Dirac system in $\mathbb{R}^3$.

Finally, we point out that in view of the non-selfadjoint nature of the problem (1.2), a study of the distribution of transmission eigenvalues in the complex plane would be both interesting and natural. So far, only in the case when $P_0 = -\Delta$, the existence of complex transmission eigenvalues has been shown in the very recent paper [4], assuming that the index of refraction is constant and sufficiently close to one. See also the numerical computation of complex transmission eigenvalues given in [11].

The structure of the paper is as follows. In Section 2 we formulate the problem and state a compact embedding lemma that is needed when proving the discreteness of the set of transmission eigenvalues in Section 3, which then follows by analytic Fredholm theory. In Section 4 sufficient conditions for the existence of transmission eigenvalues are obtained, and the existence of an infinite set of transmision eigenvalues is studied in Section 5. In Section 6 we study the reformulation of the problem which is modeled on the reduced wave, rather than the Schrödinger, equation. It is interesting that this problem poses fewer restrictions on the operator than the Schrödinger version, and thus even in the second order case we do not need to assume the ellipticity. Section 7 is devoted to the study of two examples from physics, namely, the biharmonic operator and Dirac system. In the last two Sections 8 and 9 we explain the relation of transmission eigenvalues to scattering theory, relying upon a generalized Rellich theorem.

2. INTERIOR TRANSMISSION EIGENVALUES: THE SCHRÖDINGER CASE

As described in the introduction, let $V \in L^\infty(\mathbb{R}^n)$ be compactly supported with $\mathrm{supp}\,(V) = \overline{\Omega}$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain, and let $P_0(D)$ be a partial differential operator of order $m \geq 2$ with constant real coefficients,

$$P_0(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha, \quad a_\alpha \in \mathbb{R}, \quad D_j = -i\frac{\partial}{\partial x_j}, \quad j = 1, \ldots, n.$$

The following estimate [16, Theorem 2.1], [15, Theorem 10.3.7]

$$\|\varphi\| \leq C_\Omega \|P_0(D)\varphi\|, \quad \varphi \in C_0^\infty(\Omega),$$

where C_Ω is a constant depending on the domain Ω, implies that (1.1) is a norm on $C_0^\infty(\Omega)$, and as already mentioned, completion of $C_0^\infty(\Omega)$ with respect to the norm (1.1) is denoted by $H_0^{P_0}(\Omega)$; see also [18].

Consider the interior transmission problem

$$\begin{aligned}
(P_0 - \lambda)v &= 0 \quad \text{in} \quad \Omega, \\
(P_0 + V - \lambda)w &= 0 \quad \text{in} \quad \Omega, \\
v - w &\in H_0^{P_0}(\Omega).
\end{aligned} \tag{2.1}$$

We say that $\lambda \in \mathbb{C}$ is a *transmission eigenvalue* if the problem (2.1) has non-trivial solutions $0 \neq v \in L^2_{\mathrm{loc}}(\Omega)$ and $0 \neq w \in L^2_{\mathrm{loc}}(\Omega)$. It suffices to require that $v \neq 0$. Indeed, assume that $w = 0$. Then $v \in H_0^{P_0}(\Omega)$ and extending v by zero to the complement of Ω, we get $(P_0 - \lambda)v = 0$ in $\mathbb{R}^n$. Since every constant coefficient differential operator is injective $\mathcal{E}'(\mathbb{R}^n) \to \mathcal{E}'(\mathbb{R}^n)$, we have $v = 0$.

Throughout this work we shall assume that V is real-valued and such that $V \geq \delta > 0$ a.e. in $\overline{\Omega}$.

We will use also the following equivalent characterization of the space $H_0^{P_0}(\Omega)$. Let

$$\widetilde{P}_0(\xi) = \big(\sum_{|\alpha| \geq 0} |P_0^{(\alpha)}(\xi)|^2\big)^{1/2}, \quad P_0^{(\alpha)}(\xi) = D^\alpha P_0(\xi), \quad \alpha \in \mathbb{N}^n,$$

and define on $C_0^\infty(\Omega)$ the norm

$$\|\varphi\|_{\widetilde{P}_0}^2 = \sum_{|\alpha| \geq 0} \|P_0^{(\alpha)}(D)\varphi\|^2. \tag{2.2}$$

The estimate [16, Lemma 2.8]

$$\|P_0^{(\alpha)}(D)\varphi\| \leq C_{\Omega,\alpha}\|P_0(D)\varphi\|, \quad \varphi \in C_0^\infty(\Omega), \tag{2.3}$$

yields that the norms (1.1) and (2.2) are equivalent on $H_0^{P_0}(\Omega)$

The space $H_0^{P_0}(\Omega)$ is isometrically imbedded into the space $B_{2,\widetilde{P}_0}(\mathbb{R}^n)$ via zero extension to the complement of Ω. Here

$$B_{2,\widetilde{P}_0}(\mathbb{R}^n) = \{u \in S'(\mathbb{R}^n) : \widetilde{P}_0\widehat{u} \in L^2(\mathbb{R}^n)\}$$

is a Banach space with the norm

$$u \mapsto \|\widetilde{P}_0\widehat{u}\|_{L^2}$$

introduced in [15, Definition 10.1.1]. The proof of the discreteness of the set of transmission eigenvalues depends on the following result.

Lemma 2.1. *Assume that $\widetilde{P}_0(\xi) \to \infty$ when $|\xi| \to \infty$. Then the imbedding*

$$H_0^{P_0}(\Omega) \hookrightarrow L^2(\Omega)$$

is compact.

Proof. The claim follows immediately from [15, Theorem 10.1.10].

$\square$

A condition under which we can apply this result is found with the help of the set

$$\Lambda(P_0) = \{\xi \in \mathbb{R}^n : \lambda \mapsto P_0(\lambda\xi + \eta) \text{ is constant } \forall \eta \in \mathbb{R}^n\} \tag{2.4}$$

Then [15, Proposition 10.2.9] implies that $\widetilde{P}_0(\xi) \to \infty$ as $|\xi| \to \infty$ provided that $\Lambda(P_0) = \{0\}$, i.e. there are no hidden variables in $P_0(\xi)$.

3. Discreteness of the set of transmission eigenvalues

Proposition 3.1. *A complex number λ is a transmission eigenvalue if and only if there exists $0 \neq u \in H_0^{P_0}(\Omega)$ satisfying*

$$(P_0 + V - \lambda)\frac{1}{V}(P_0 - \lambda)u = 0 \quad in \quad \mathcal{D}'(\Omega). \tag{3.1}$$

Proof. Assume that λ is a transmission eigenvalue. Then there exists a non-trivial solution $(v, w) \neq 0$ of (2.1). Define $u = v - w \in H_0^{P_0}(\Omega)$. Then $(P_0 - \lambda)u = Vw$ and thus,

$$(P_0 + V - \lambda)\frac{1}{V}(P_0 - \lambda)u = (P_0 + V - \lambda)w = 0.$$

To see the reverse implication, we first claim that

$$(P_0 + V - \lambda)\frac{1}{V}(P_0 - \lambda) = (P_0 - \lambda)\frac{1}{V}(P_0 + V - \lambda).$$

Indeed,

$$(P_0 + V - \lambda)\frac{1}{V}(P_0 - \lambda)u - (P_0 - \lambda)\frac{1}{V}(P_0 + V - \lambda)u$$

$$= (P_0 - \lambda)\frac{1}{V}(P_0 - \lambda)u + \frac{V}{V}(P_0 - \lambda)u - (P_0 - \lambda)\frac{1}{V}(P_0 - \lambda)u$$

$$- (P_0 - \lambda)\frac{V}{V}u = 0.$$

Let $0 \neq u \in H_0^{P_0}(\Omega)$ be a solution of (3.1). Denote now

$$w = \frac{1}{V}(P_0 - \lambda)u \quad \text{and} \quad v = \frac{1}{V}(P_0 + V - \lambda)u.$$

Then

$$(P_0 - \lambda)v = 0, \quad (P_0 + V - \lambda)w = 0 \quad in \quad \Omega$$

and

$$v - w = \frac{1}{V}(P_0 + V - \lambda - P_0 + \lambda)u = u \in H_0^{P_0}(\Omega).$$

$\square$

Set

$$T_\lambda = (P_0 + V - \lambda)\frac{1}{V}(P_0 - \lambda).$$

Let $\lambda \in \mathbb{C}$. Then for $\varphi, \psi \in C_0^\infty(\Omega)$, we define a sesquilinear form

$$B_\lambda(\varphi, \psi) = \langle T_\lambda \varphi, \psi \rangle_{L^2(\Omega)} = \langle \frac{1}{V}(P_0 - \lambda)\varphi, (P_0 + V - \lambda)\psi \rangle_{L^2(\Omega)}$$

which extends uniquely to a continuous sesquilinear form on $H_0^{P_0}(\Omega) \times H_0^{P_0}(\Omega)$.

Lemma 3.2. *Assume that either $P_0(\xi)$ or $-P_0(\xi)$ is bounded from below on $\mathbb{R}^n$. Then there exists $\lambda_0 \in \mathbb{R}$ such that the sesquilinear form $B_{\lambda_0}(u, v)$ is coercive on $H_0^{P_0}(\Omega) \times H_0^{P_0}(\Omega)$.*

Proof. Let $\varphi \in C_0^\infty(\Omega)$ and set $C_0 = \inf_\Omega 1/V > 0$. Then

$$
\begin{aligned}
B_\lambda(\varphi, \varphi) &= \langle (P_0 - \lambda)\frac{1}{V}(P_0 - \lambda)\varphi, \varphi \rangle_{L^2(\Omega)} + \langle (P_0 - \lambda)\varphi, \varphi \rangle_{L^2(\Omega)} \\
&\geq C_0 \langle (P_0 - \lambda)^2 \varphi, \varphi \rangle_{L^2(\Omega)} + \langle (P_0 - \lambda)\varphi, \varphi \rangle_{L^2(\Omega)} \\
&= C_0 \|P_0 \varphi\|^2 - 2C_0 \lambda \langle P_0 \varphi, \varphi \rangle_{L^2(\Omega)} \\
&\quad + \lambda^2 C_0 \|\varphi\|^2 - \lambda \|\varphi\|^2 + \langle P_0 \varphi, \varphi \rangle_{L^2(\Omega)}.
\end{aligned}
\tag{3.2}
$$

To estimate the last term in the right hand side of (3.2) we write

$$
\langle P_0 \varphi, \varphi \rangle_{L^2(\Omega)} \leq \|P_0 \varphi\| \|\varphi\| \leq \frac{C_0}{2} \|P_0 \varphi\|^2 + \frac{2}{C_0} \|\varphi\|^2.
\tag{3.3}
$$

For the second term in the right hand side of (3.2), assuming, to fix the ideas, that $P_0(\xi)$ is bounded from below on $\mathbb{R}^n$, we get

$$
\langle P_0 \varphi, \varphi \rangle_{L^2(\Omega)} = (2\pi)^{-n} \int_{\mathbb{R}^n} P_0(\xi) |\widehat{\varphi}(\xi)|^2 d\xi \geq -C \|\varphi\|^2.
\tag{3.4}
$$

Now combining (3.2), (3.3) and (3.4), we can find $\lambda_0 \in \mathbb{R}$ with $|\lambda_0|$ large enough such that

$$
B_{\lambda_0}(\varphi, \varphi) \geq \frac{C_0}{2} \|P_0 \varphi\|^2 = \frac{C_0}{2} \|\varphi\|_{P_0}^2, \quad \forall \varphi \in C_0^\infty(\Omega).
$$

This establishes the coercivity of B_{λ_0} on $H_0^{P_0}(\Omega) \times H_0^{P_0}(\Omega)$.

$\square$

Let us denote by $H^{-P_0}(\Omega)$ the dual of the space $H_0^{P_0}(\Omega)$. Notice that $H^{-P_0}(\Omega)$ can be viewed as a subspace of the space of distributions $\mathcal{D}'(\Omega)$.

Theorem 3.3. *Assume that $\widetilde{P_0}(\xi) \to \infty$ when $|\xi| \to \infty$ and that the symbol $\pm P_0(\xi)$ is bounded from below on $\mathbb{R}^n$, for one of the choices of the sign. Then the set of transmission eigenvalues is discrete.*

Proof. The proof is based on application of the analytic Fredholm theory to the holomorphic family of operators

$$
T_\lambda : \mathbb{C} \to \mathcal{L}(H_0^{P_0}(\Omega), H^{-P_0}(\Omega)), \quad T_\lambda = (P_0 + V - \lambda)\frac{1}{V}(P_0 - \lambda).
$$

Since by Lemma 3.2 there exists $\lambda_0 \in \mathbb{R}$ such that the sesquilinear form B_{λ_0}, corresponding to the operator T_{λ_0}, is coercive and bounded on $H_0^{P_0}(\Omega) \times H_0^{P_0}(\Omega)$, by an application of the Lax–Milgram lemma we conclude that $T_{\lambda_0} : H_0^{P_0}(\Omega) \to H^{-P_0}(\Omega)$ is invertible.

Now, for any λ the operator

$$T_\lambda - T_{\lambda_0} = -P_0 \frac{\lambda - \lambda_0}{V} - \frac{\lambda - \lambda_0}{V} P_0 - (\lambda - \lambda_0) + \frac{\lambda^2 - \lambda_0^2}{V} : H_0^{P_0}(\Omega) \to H^{-P_0}(\Omega)$$
(3.5)

is compact. Indeed, the last two terms in the right hand side of (3.5) are compact because

$$-(\lambda - \lambda_0) + \frac{\lambda^2 - \lambda_0^2}{V} : L^2(\Omega) \to L^2(\Omega)$$

is bounded and by Lemma 2.1 the embedding $H_0^{P_0}(\Omega) \hookrightarrow L^2(\Omega)$ is compact. The second term in the right hand side of (3.5) defines the operator $-\frac{\lambda - \lambda_0}{V} P_0$. This operator is compact as a composition of the bounded operators

$$P_0 : H_0^{P_0}(\Omega) \to L^2(\Omega), \quad -\frac{\lambda - \lambda_0}{V} : L^2(\Omega) \to L^2(\Omega)$$

with the compact inclusion $L^2(\Omega)$ into $H^{-P_0}(\Omega)$. The first term in the right hand side of (3.5) defines the operator $-P_0 \frac{\lambda - \lambda_0}{V}$. This is a bounded operator

$$-P_0 \frac{\lambda - \lambda_0}{V} : L^2(\Omega) \to H^{-P_0}(\Omega).$$

Since the inclusion $H_0^{P_0}(\Omega)$ to $L^2(\Omega)$ is compact, we conclude the compactness of $-P_0 \frac{\lambda - \lambda_0}{V}$ as an operator from $H_0^{P_0}(\Omega)$ to $H^{-P_0}(\Omega)$. Thus also $T_\lambda - T_{\lambda_0}$ is compact.

Hence, the operator T_λ is Fredholm as the sum of an invertible operator and a compact operator. Since it is invertible at $\lambda = \lambda_0$, the analytic Fredholm theory guarantees that T_λ is invertible except for a discrete set of values λ.

$\square$

Remark 3.1. The assumptions of Theorem 3.3 are satisfied automatically when the operator P_0 is elliptic.

Remark 3.2. The condition that $\pm P_0(\xi)$ is bounded from below on $\mathbb{R}^n$, for one of the choices of the sign, cannot be removed completely. Indeed, let $P_0 = D_{x_1} + \Delta_{x'}$, $x = (x_1, x') \in \mathbb{R}^n$. Then if (v, w) is a pair of transimssion eigenfunctions for a transmission eigenvalue λ on some domain Ω, then $v'(x) = e^{i\mu x_1} v(x)$ and $w'(x) = e^{i\mu x_1} w(x)$ will be transmission eigenfunctions for the transmission eigenvalue $\lambda + \mu$. Hence the set of transmission eigenvalues is either empty or $\mathbb{C}$.

Remark 3.3. Notice that $\mathrm{ind}(T_\lambda) = 0$, since T_λ is the sum of an invertible operator and a compact operator. It follows that T_λ fails to be invertible if and only if T_λ is not injective.

4. EXISTENCE OF TRANSMISSION EIGENVALUES

In this section we study the question of existence of transmission eigenvalues. Let us write the operator T_λ in the following form,

$$T_\lambda = (P_0 + \frac{V}{2} - \lambda)\frac{1}{V}(P_0 + \frac{V}{2} - \lambda) - \frac{V}{4}. \tag{4.1}$$

Lemma 4.1. *The operator T_λ given by* (4.1)*, equipped with the domain*

$$\mathcal{D}(T_\lambda) = \{u \in H_0^{P_0}(\Omega) : P_0(\frac{1}{V}P_0 - \frac{\lambda}{V})u \in L^2(\Omega)\},$$

is an unbounded self-adjoint operator on $L^2(\Omega)$, for any $\lambda \in \mathbb{R}$.

Proof. Let $A_\lambda = \frac{1}{V^{1/2}}(P_0 + \frac{V}{2} - \lambda)$. Then A_λ with the domain $\mathcal{D}(A_\lambda) = H_0^{P_0}(\Omega)$ is a densely defined closed operator. Its L^2–adjoint is given by $A_\lambda^* = (P_0 + \frac{V}{2} - \lambda)\frac{1}{V^{1/2}}$ with the domain $\mathcal{D}(A_\lambda^*) = \{u \in L^2(\Omega) : P_0\frac{1}{V^{1/2}}u \in L^2(\Omega)\}$. Then the result follows from the fact that $T_\lambda = A_\lambda^* A_\lambda - \frac{V}{4}$.   □

For future reference, let us remark that the form domain of the semibounded self-adjoint operator T_λ is $H_0^{P_0}(\Omega)$, for any $\lambda \in \mathbb{R}$. Let us also recall the corresponding quadratic form

$$B_\lambda(u, u) = \langle \frac{1}{V}(P_0 - \lambda)u, (P_0 + V - \lambda)u \rangle_{L^2(\Omega)}, \quad u \in H_0^{P_0}(\Omega).$$

From Lemma 3.2, we know that there exists $\lambda_0 \in \mathbb{R}$ such that B_{λ_0} is coercive on $H_0^{P_0}(\Omega)$.

Similarly to [23], we have the following result.

Theorem 4.2. *If for some $\lambda \in \mathbb{R}$, there is $u \in H_0^{P_0}(\Omega)$ such that*

$$B_\lambda(u, u) \leq 0 \tag{4.2}$$

then there exists a transmission eigenvalue $\lambda^* \in (\lambda_0, \lambda]$, if $\lambda_0 < \lambda$, or $\lambda^* \in [\lambda, \lambda_0)$, if $\lambda_0 > \lambda$.

Proof. The operator T_λ has a compact resolvent acting on $L^2(\Omega)$ because the canonical embedding of its domain $\mathcal{D}(T_\lambda)$ to $L^2(\Omega)$ is compact. The facts that T_λ depends continuously on λ, T_λ is self-adjoint with compact resolvent and the form domain of T_λ does not depend on λ imply, through an application of the variational principle, that the eigenvalues of T_λ depend continuously on λ for $\lambda \in \mathbb{R}$. See [23] for the details of this argument.

The hypothesis (4.2) yields that T_λ has at least one non-positive eigenvalue. Since T_{λ_0} is positive definite, the lowest eigenvalue, which is a continuous function of $\lambda \in \mathbb{R}$, must pass through zero for some $\lambda^* \in (\lambda_0, \lambda]$, if $\lambda_0 < \lambda$, or $\lambda^* \in [\lambda, \lambda_0)$, if

$\lambda_0 > \lambda$. Hence, the operator T_λ^* is non-injective and therefore, λ^* is a transmission eigenvalue.

$\square$

We define the multiplicity of a transmission eigenvalue λ to be the multiplicity of zero as an eigenvalue of T_λ. Since the self-adjoint operator T_λ, $\lambda \in \mathbb{R}$, has a compact resolvent, the multiplicity of λ is finite.

As in [23, Lemma 14], we get the following result on existence of more than one transmission eigenvalues.

Lemma 4.3. *If there exists $\lambda \in \mathbb{R}$ and a p-dimensional subspace $V^p \subset H_0^{P_0}(\Omega)$ such that*

$$B_\lambda(u, u) \leq 0 \tag{4.3}$$

for all $u \in V^p$, then there exist p transmission eigenvalues, counting with multiplicity.

The next result tells us that if the potential is strong enough then the transmission eigenvalues exist.

Lemma 4.4. *Let $V(x) \geq \delta > 0$. If there exists a p-dimensional subspace $V^p \subset H_0^{P_0}(\Omega)$ such that*

$$\delta \geq 2\frac{\|P_0 u\|}{\|u\|}$$

for all $0 \neq u \in V^p$, then there exist p transmission eigenvalues, counting with multiplicity.

Proof. We have

$$B_\lambda(u, u) = \langle \frac{1}{V}(P_0 - \lambda)u, (P_0 - \lambda)u \rangle + \langle (P_0 - \lambda)u, u \rangle$$

$$\leq \frac{1}{\delta}\|P_0 u\|^2 - \frac{2\lambda}{\delta}\langle P_0 u, u \rangle + \frac{\lambda^2}{\delta}\|u\|^2 + \langle P_0 u, u \rangle - \lambda\|u\|^2.$$

Choosing $\lambda = \delta/2$, we get

$$B_\lambda(u, u) \leq \frac{1}{\delta}\|P_0 u\|^2 - \frac{\delta}{4}\|u\|^2.$$

The claim follows by an application of Lemma 4.3. $\square$

5. Existence of infinitely many transmission eigenvalues

Now assume, as before, that $P_0(D)$ is a partial differential operator of degree $m \geq 2$ with constant real coefficients and $\widetilde{P}_0(\xi) \to \infty$ when $|\xi| \to \infty$ and the symbol $\pm P_0(\xi)$ is bounded from below on $\mathbb{R}^n$, for one of the choices of the sign. Let $\varepsilon > 0$ and consider an open ball $B_\varepsilon(0) \subset \mathbb{R}^n$ of radius ε centered at the origin.

Consider the following interior transmission problem for the ball $B_\varepsilon(0)$ and an arbitrary constant potential $\delta > 0$,

$$
\begin{aligned}
(P_0 - \lambda)v &= 0 \quad \text{in} \quad B_\varepsilon(0), \\
(P_0 + \delta - \lambda)w &= 0 \quad \text{in} \quad B_\varepsilon(0), \\
v - w &\in H_0^{P_0}(B_\varepsilon(0)).
\end{aligned}
\tag{5.1}
$$

Notice that as the operator P_0 has constant coefficients and δ is a constant, if there exists a transmission eigenvalue $\lambda(\varepsilon)$ for the ball $B_\varepsilon(0)$, then $\lambda(\varepsilon)$ is a transmission eigenvalue for an arbitrary ball in $\mathbb{R}^n$ of radius ε.

Existence of an infinite discrete set of real transmission eigenvalues when $P_0 = -\Delta$ in $\mathbb{R}^n$, $n = 2, 3$, was obtained in [5]. The proof relies on an explicit computation in the case of a constant potential when Ω is a ball, combined with a specific analytic framework of generalized eigenvalue problems.

Following the approach of [5], we have the following conditional result on existence of an infinite set of real transmission eigenvalues for general operators. In the proof we give a direct argument relying upon Lemma 4.3.

Proposition 5.1. *Let $V \in L^\infty(\mathbb{R}^n)$ be real-valued compactly supported with $\mathrm{supp}\,(V) = \overline{\Omega}$ and assume that*

$$
V \geq \delta > 0 \quad \text{a.e. in } \overline{\Omega}.
$$

Furthermore, assume that for any $\varepsilon > 0$, there exists a real transmission eigenvalue for (5.1). Then the problem (2.1) has an infinite set of real transmission eigenvalues.

Proof. For every $p \in \mathbb{N}$, there exists $\varepsilon > 0$ small enough such that Ω contains p disjoint balls $B_\varepsilon^1, \ldots, B_\varepsilon^p$ of radius ε, i.e. $\overline{B_\varepsilon^i} \subset \Omega$, $i = 1, \ldots, p$, and $\overline{B_\varepsilon^i} \cap \overline{B_\varepsilon^j} = \emptyset$ for $i \neq j$. Let $\lambda(\varepsilon) \in \mathbb{R}$ be a transmission eigenvalue for (5.1). Then it is a transmission eigenvalue for each of the balls B_ε^i with potential δ. Thus, Proposition 3.1 implies that there are $0 \neq u^i(\varepsilon) \in H_0^{P_0}(B_\varepsilon^i)$ such that

$$
(P_0 + \delta - \lambda(\epsilon))\frac{1}{\delta}(P_0 - \lambda)u^i(\varepsilon) = 0 \quad \text{in} \quad \mathcal{D}'(B_\varepsilon^i), \quad i = 1, \ldots, p.
$$

The extension by zero $\widetilde{u}^i$ of $u^i(\varepsilon)$ to the whole of Ω is in $H_0^{P_0}(\Omega)$. Moreover, the functions $\widetilde{u}^1, \ldots, \widetilde{u}^p$ form an orthogonal system in $H_0^{P_0}(\Omega)$, since they have disjoint supports. This implies that

$$
\langle \frac{1}{\delta}(P_0 - \lambda(\varepsilon))\widetilde{u}^i, (P_0 + \delta - \lambda(\varepsilon))\widetilde{u}^j \rangle_{L^2(\Omega)} = 0, \quad i, j = 1, \ldots, p.
$$

Set $V^p = \text{span}\{\widetilde{u}^1, \ldots, \widetilde{u}^p\}$. Let $u \in V^p$. Then

$$B_{\lambda(\varepsilon)}(u, u) = \langle \frac{1}{V}(P_0 - \lambda(\varepsilon))u, (P_0 + V - \lambda(\varepsilon))u \rangle_{L^2(\Omega)}$$

$$= \langle \frac{1}{V}(P_0 - \lambda(\varepsilon))u, (P_0 - \lambda(\varepsilon))u \rangle_{L^2(\Omega)} + \langle \frac{1}{V}(P_0 - \lambda(\varepsilon))u, Vu \rangle_{L^2(\Omega)}$$

$$\leq \langle \frac{1}{\delta}(P_0 - \lambda(\varepsilon))u, (P_0 - \lambda(\varepsilon))u \rangle_{L^2(\Omega)} + \langle (P_0 - \lambda(\varepsilon))u, u \rangle_{L^2(\Omega)}$$

$$= \langle \frac{1}{\delta}(P_0 - \lambda(\varepsilon))u, (P_0 + \delta - \lambda(\varepsilon))u \rangle_{L^2(\Omega)} = 0.$$

Hence, it follows from Lemma 4.3 that problem (2.1) has p transmission eigenvalues, counting with multiplicity. As p is arbitrary, the result follows.

$\square$

6. The generalized acoustic problem

Let $V \in L^\infty(\mathbb{R}^n)$ be compactly supported in $\mathbb{R}^n$ with $\text{supp}\,(V) = \overline{\Omega}$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain, and $P_0(D)$ be a partial differential operator of degree $m \geq 2$ with constant real coefficients.

As in physics, while the problem

$$(-\Delta - \lambda + V)u = 0,$$

models the time-independent Schrödinger equation, the equation

$$(-\Delta - \lambda(1 + V))u = 0$$

describes acoustic wave propagation with refractive index $1 + V$. In this section, we study the interior transmission problem for the latter, where $-\Delta$ is replaced by a general P_0. This problem has the following form,

$$
\begin{aligned}
(P_0 - \lambda)v &= 0 \quad \text{in} \quad \Omega, \\
(P_0 - \lambda(1 + V))w &= 0 \quad \text{in} \quad \Omega, \\
v - w &\in H_0^{P_0}(\Omega).
\end{aligned}
\tag{6.1}
$$

We say that $\lambda \in \mathbb{C}$ is a transmission eigenvalue if the problem (6.1) has non-trivial solutions $0 \neq v \in L^2_{\text{loc}}(\Omega)$ and $0 \neq w \in L^2_{\text{loc}}(\Omega)$. It suffices to require that $v \neq 0$. Notice that $\lambda = 0$ is always a transmission eigenvalue for (6.1).

We shall assume that V is real-valued and such that $V \geq \delta > 0$ a.e. in $\overline{\Omega}$.

As in Proposition 3.1, one can show that $0 \neq \lambda \in \mathbb{C}$ is a transmission eigenvalue if and only if there exists $0 \neq u \in H_0^{P_0}(\Omega)$ satisfying

$$T_\lambda u = (P_0 - \lambda(1 + V))\frac{1}{V}(P_0 - \lambda)u = 0 \quad \text{in} \quad \mathcal{D}'(\Omega).$$

Define a sesquilinear form

$$B_\lambda(\varphi, \psi) = \langle T_\lambda \varphi, \psi \rangle_{L^2(\Omega)} = \langle \frac{1}{V}(P_0 - \lambda)\varphi, (P_0 - \lambda(1+V))\psi \rangle_{L^2(\Omega)}, \quad \varphi, \psi \in C_0^\infty(\Omega),$$

which extends uniquely to a continuous sesquilinear for on $H_0^{P_0}(\Omega) \times H_0^{P_0}(\Omega)$. Then B_0 is coercive in the sense that there exists $C_0 > 0$ such that

$$B_0(\varphi, \varphi) \geq C_0 \|\varphi\|_{P_0}^2, \quad \varphi \in H_0^{P_0}(\Omega).$$

Arguing as in Theorem 3.3, we get the following result.

Theorem 6.1. *Assume that $\widetilde{P}_0(\xi) \to \infty$ when $|\xi| \to \infty$. Then the set of transmission eigenvalues is discrete.*

Remark that the multiplicity of each transmission eigenvalue except $\lambda = 0$ is finite.

Notice that the assumption in Theorem 3.3 that the symbol $\pm P_0(\xi)$ is bounded from bellow on $\mathbb{R}^n$, for one of the choices of the sign, is not needed in the Helmholtz case. In particular, the results of this section are applicable when $P_0 = D_{x_1}^2 + \Delta_{x'}$ is the wave operator on $\mathbb{R}^n$.

An inspection of the arguments of Sections $4 - 5$ shows that the results there remain valid in the Helmholtz case. In particular, Theorem 4.2 and Lemma 4.3 hold true in the Helmholtz case as they stand. To formulate an analog of Proposition 5.1, consider the following interior transmission problem for an open ball $B_\varepsilon(0) \subset \mathbb{R}^n$ of radius $\varepsilon > 0$ centered at the origin, and an arbitrary constant potential $\delta > 0$,

$$\begin{aligned}
(P_0 - \lambda)v &= 0 \quad \text{in} \quad B_\varepsilon(0), \\
(P_0 - \lambda(1 + \delta))w &= 0 \quad \text{in} \quad B_\varepsilon(0), \\
v - w &\in H_0^{P_0}(B_\varepsilon(0)).
\end{aligned} \tag{6.2}$$

Proposition 6.2. *Assume that for any $\varepsilon > 0$ and $\delta > 0$, there exists a non-zero real transmission eigenvalue for (6.2). Then there exists an infinite set of real transmission eigenvalues for (6.1).*

7. Examples

7.1. **The biharmonic operator.** As an example of a higher order operator to which our conditional existence results apply, we shall consider the biharmonic operator in $\mathbb{R}^3$, which arises, e.g., in the study of thin elastic plates.

Proposition 7.1. *Let $P_0 = \Delta^2$ on $\mathbb{R}^3$. Then the problem (6.1) has an infinite set of real transmission eigenvalues.*

Proof. When proving this result, we shall apply Proposition 6.2. It therefore suffices to prove that the problem (6.2) has a non-trivial solution for arbitrary $\varepsilon > 0$ and $\delta > 0$. Without loss of generality, we may assume that δ is sufficiently small. Let $k = \lambda^{1/4}$, $k_\delta = (\lambda(1+\delta))^{1/4}$, and

$$\rho = \frac{k_\delta}{k} = (1+\delta)^{1/4} = 1 + \frac{\delta}{4} + \mathcal{O}(\delta^2), \quad 0 < \delta \ll 1. \tag{7.1}$$

Then writing

$$\Delta^2 - k^4 = (\Delta - k^2)(\Delta + k^2) = (\Delta + k^2)(\Delta - k^2),$$

and for $\Delta^2 - k_\delta^4$ similarly, one sees that a reasonable ansatz for v and w in (6.2) is

$$v(x) = a_1 j_0(kr) + a_2 j_0(ikr), \ \ w(x) = a_3 j_0(k_\delta r) + a_4 j_0(ik_\delta r),$$

where j_0 is the spherical Bessel function of order zero, and a_j are constants. Our boundary conditions now take form

$$\left(\frac{d}{dr}\right)^l (a_1 j_0(kr) + a_2 j_0(ikr) - a_3 j_0(k_\delta r) - a_4 j_0(ik_\delta r))|_{r=\varepsilon} = 0, \quad l = 0, \ 1, \ 2, \ 3.$$
$$\tag{7.2}$$

Using the known asymptotics for $j_0(k\varepsilon)$ and $j_0(ik\varepsilon)$ and their derivatives, as $k \to \infty$, one sees that the deteminant of the linear system (7.2) is

$$d = \begin{vmatrix} \frac{\sin(k\varepsilon)}{k\varepsilon} & \frac{e^{k\varepsilon}}{2k\varepsilon} + \mathcal{O}(e^{-k\varepsilon}) & -\frac{\sin(k_\delta\varepsilon)}{k_\delta\varepsilon} & -\frac{e^{k_\delta\varepsilon}}{2k_\delta\varepsilon} + \mathcal{O}(e^{-k_\delta\varepsilon}) \\ \frac{\cos(k\varepsilon)}{\varepsilon} + \mathcal{O}(\frac{1}{k}) & \frac{e^{k\varepsilon}}{2\varepsilon} + \mathcal{O}(\frac{e^{k\varepsilon}}{k}) & -\frac{\cos(k_\delta\varepsilon)}{\varepsilon} + \mathcal{O}(\frac{1}{k}) & -\frac{e^{k_\delta\varepsilon}}{2\varepsilon} + \mathcal{O}(\frac{e^{k_\delta\varepsilon}}{k_\delta}) \\ -\frac{k\sin(k\varepsilon)}{\varepsilon} + \mathcal{O}(1) & \frac{ke^{k\varepsilon}}{2\varepsilon} + \mathcal{O}(e^{k\varepsilon}) & \frac{k_\delta\sin(k_\delta\varepsilon)}{\varepsilon} + \mathcal{O}(1) & -\frac{k_\delta e^{k_\delta\varepsilon}}{2\varepsilon} + \mathcal{O}(e^{k_\delta\varepsilon}) \\ -\frac{k^2\cos(k\varepsilon)}{\varepsilon} + \mathcal{O}(k) & \frac{k^2 e^{k\varepsilon}}{2\varepsilon} + \mathcal{O}(ke^{k\varepsilon}) & \frac{k_\delta^2\cos(k_\delta\varepsilon)}{\varepsilon} + \mathcal{O}(k) & -\frac{k_\delta^2 e^{k_\delta\varepsilon}}{2\varepsilon} + \mathcal{O}(k_\delta e^{k_\delta\varepsilon}) \end{vmatrix}$$

$$= \frac{1}{4\varepsilon^4} e^{(k+k_\delta)\varepsilon} k^2 \frac{\rho - 1}{\rho}\left(d_1 + \mathcal{O}(\frac{1}{\delta k})\right),$$

where d_1 is given by

$$\begin{vmatrix} (\rho - 1)\sin(k\varepsilon) - (1+\rho)\cos(k\varepsilon) & (\rho - 1)\sin(k_\delta\varepsilon) + (\rho + 1)\cos(k_\delta\varepsilon) \\ -(\rho^2 + \rho + 2)\cos(k\varepsilon) + (\rho^2 + \rho)\sin(k\varepsilon) & (2\rho^2 + \rho + 1)\cos(k_\delta\varepsilon) - (\rho + 1)\sin(k_\delta\varepsilon) \end{vmatrix}$$

Using (7.1), we get

$$d_1 = 4\sin((\rho - 1)k\varepsilon) + \mathcal{O}(\delta).$$

Hence, to show that the linear system (7.2) has a non-trivial solution, it suffices to check that the function

$$\sin((\rho - 1)k\varepsilon) + \mathcal{O}(\delta) + \mathcal{O}(\frac{1}{\delta k})$$

has real zeros, for k large enough. The latter is clear, however, from the periodicity of the function

$$k \mapsto \sin((\rho - 1)k\varepsilon).$$

We may also notice that the minimal period of this function is

$$\frac{8\pi}{\delta\varepsilon} + \mathcal{O}(\frac{1}{\varepsilon}).$$

This completes the proof.

$\square$

7.2. **The Dirac system.** Our approach generalizes also to many systems. We demonstrate this here by carrying out the analysis in the case of the Dirac system.

The free Dirac operator in $\mathbb{R}^3$ is given by the 4×4 matrix

$$\mathcal{L}_0(D) = \begin{pmatrix} 0 & \sigma \cdot D \\ \sigma \cdot D & 0 \end{pmatrix},$$

where $D = -i\nabla$ and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is a vector of Pauli matrices with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The most important property of the Dirac operator is the following one,

$$\mathcal{L}_0(D)^2 = -\Delta I_4,$$

where I_4 is the 4×4 identity matrix.

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain in $\mathbb{R}^3$ with a connected C^∞-smooth boundary. It is known [22] that when equipped with the domain

$$\mathcal{D}(\mathcal{L}_0) = \{ \begin{pmatrix} u_+ \\ u_- \end{pmatrix} \in L^2(\Omega)^2 \times L^2(\Omega)^2 : \ u_+ \in H^1_0(\Omega)^2, \sigma \cdot Du_- \in L^2(\Omega)^2 \},$$

the Dirac operator $\mathcal{L}_0$ is self-adjoint on $L^2(\Omega)^4$.

Notice that

$$H^1_0(\Omega)^2 \times H^1(\Omega)^2 \subset \mathcal{D}(\mathcal{L}_0).$$

However, in general, $\mathcal{D}(\mathcal{L}_0)$ is strictly larger than the Sobolev space $H^1_0(\Omega)^2 \times H^1(\Omega)^2$, see [25] for the discussion and a precise example.

Let $V(x)$ be an Hermitian 4×4-matrix-valued function whose entries belong to $L^\infty(\mathbb{R}^3)$. An application of the Kato-Rellich theorem shows that the operator $\mathcal{L}_0(D) + V$ is self-adjoint on $\mathcal{D}(\mathcal{L}_0)$. Assume that 0 is not in the spectrum of $\mathcal{L}_0(D) + V$. Then it was shown in [22, 24] that for any $f \in H^{1/2}(\partial\Omega)^2$, the boundary value problem

$$\begin{aligned} (\mathcal{L}_0(D) + V)u &= 0, \quad \text{in} \quad \Omega, \\ u_+ &= f, \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{7.3}$$

has a unique solution $u \in H^1(\Omega)^4$. The set of the Cauchy data for (7.3) is given by

$$\{(u_+|_{\partial\Omega}, u_-|_{\partial\Omega}) : u \in H^1(\Omega)^4 \text{ is a solution of } (\mathcal{L}_0(D) + V)u = 0 \text{ in } \Omega\}.$$

Assume now that supp $(V) = \overline{\Omega}$. The interior transmission problem for the Dirac operator is the following boundary value problem,

$$
\begin{aligned}
(\mathcal{L}_0(D) - \lambda I_4)v &= 0 \quad \text{in} \quad \Omega, \\
(\mathcal{L}_0(D) - \lambda(I_4 + V))w &= 0 \quad \text{in} \quad \Omega, \\
v - w &\in H_0^1(\Omega)^4.
\end{aligned}
\tag{7.4}
$$

We say that $\lambda \in \mathbb{C}$ is a transmission eigenvalue if the problem (7.4) has non-trivial solutions $0 \neq v \in L_{\text{loc}}^2(\Omega)^4$ and $0 \neq w \in L_{\text{loc}}^2(\Omega)^4$. It suffices to require that $v \neq 0$.

Notice that $\lambda = 0$ is a transmission eigenvalue and the space of functions

$$
\{v \neq 0 : \mathcal{L}_0(D)v = 0\}
$$

is infinite dimensional.

Remark 7.1. The standard electromagnetic potential for the Dirac operator given by

$$
V = \mathcal{L}_0(A) + Q,
$$

with

$$
Q = \begin{pmatrix} q_+ I_2 & 0 \\ 0 & q_- I_2 \end{pmatrix}, \quad A = (a_1, a_2, a_3) \in L^\infty(\Omega; \mathbb{R}^3), \quad q_\pm \in L^\infty(\Omega; \mathbb{R})
$$

is included in the setup above.

Throughout this section, we shall assume that $V(x)$ is an Hermitian positive-definite 4×4-matrix valued function, i.e. there is a constant $c_V > 0$ such that

$$
\langle V(x)\eta, \eta \rangle \geq c_V |\eta|^2, \quad \forall x \in \overline{\Omega}, \quad \forall \eta \in \mathbb{C}^4,
$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{C}^4$. Moreover, we shall assume that the entries of $V(x)$ belong to $C^\infty(\overline{\Omega})$. Thus, the entries of the inverse matrix $V^{-1}(x)$ also belong to $C^\infty(\overline{\Omega})$.

Arguing as in the earlier sections, we see that the following characterization of transmission eigenvalues holds: $0 \neq \lambda \in \mathbb{C}$ is a transmission eigenvalue if and only if there exists $0 \neq u \in H_0^1(\Omega)^4$ satisfying

$$
T_\lambda u = (\mathcal{L}_0 - \lambda(I_4 + V))V^{-1}(\mathcal{L}_0 - \lambda I_4)u = 0 \quad \text{in} \quad \mathcal{D}'(\Omega)^4.
$$

Here

$$
T_\lambda = A - \lambda B + \lambda^2 C,
$$

where

$$
\begin{aligned}
A &= \mathcal{L}_0 V^{-1} \mathcal{L}_0, \\
B &= V^{-1} \mathcal{L}_0 + \mathcal{L}_0 V^{-1} + \mathcal{L}_0, \\
C &= 1 + V^{-1}.
\end{aligned}
$$

Proposition 7.2. *The operator A is symmetric and positive on $L^2(\Omega)^4$, when equipped with the domain $C_0^\infty(\Omega)^4$, in the sense that there is $d = d_{V,\Omega} > 0$ such that*

$$\langle A\varphi, \varphi \rangle_{L^2(\Omega)^4} \geq d\|\varphi\|_{L^2(\Omega)^4}^2, \quad \varphi \in C_0^\infty(\Omega)^4.$$

Proof. Let $\varphi \in C_0^\infty(\Omega)^4$. Then we get

$$\langle A\varphi, \varphi \rangle_{L^2(\Omega)^4} = \langle V^{-1}\mathcal{L}_0\varphi, \mathcal{L}_0\varphi \rangle_{L^2(\Omega)^4} \geq c_V \|\mathcal{L}_0\varphi\|_{L^2(\Omega)^4}^2 \geq d\|\varphi\|_{L^2(\Omega)^4}^2.$$

Here the last inequality follows from the estimate [15, Theorem 10.3.7]

$$\|\mathcal{L}_0\varphi\|_{L^2(\Omega)^4} \geq c_\Omega \|\varphi\|_{L^2(\Omega)^4}, \quad \varphi \in C_0^\infty(\Omega)^4, \quad c_\Omega > 0.$$

The claim follows.

$\square$

Proposition 7.3. *The second order operator A is uniformly strongly elliptic in the sense that there is $c > 0$ such that*

$$\langle \sigma(A)(x,\xi)\eta, \eta \rangle \geq c|\xi|^2|\eta|^2, \quad x \in \overline{\Omega}, \quad \xi \in \mathbb{R}^3, \quad \eta \in \mathbb{C}^4, \tag{7.5}$$

where $\sigma(A)$ is the principal symbol of A.

Proof. In view of homogeneity of (7.5) it suffices to prove it for $|\xi| = 1$ and $|\eta| = 1$. Let $\eta = (\eta_+, \eta_-)^T \in \mathbb{C}^4$. Then since V^{-1} is an Hermitian positive-definite matrix valued function, we have

$$\langle \sigma(A)(x,\xi)\eta, \eta \rangle = \langle V^{-1} \begin{pmatrix} 0 & \sigma \cdot \xi \\ \sigma \cdot \xi & 0 \end{pmatrix} \eta, \begin{pmatrix} 0 & \sigma \cdot \xi \\ \sigma \cdot \xi & 0 \end{pmatrix} \eta \rangle$$

$$\geq c_V \left| \begin{matrix} (\sigma \cdot \xi)\eta_- \\ (\sigma \cdot \xi)\eta_+ \end{matrix} \right|_{\mathbb{C}^4}^2 > 0.$$

The latter inequality follows from the fact that

$$\sigma \cdot \xi = \begin{pmatrix} \xi_3 & \xi_1 - i\xi_2 \\ \xi_1 + i\xi_2 & -\xi_3 \end{pmatrix}, \quad \det(\sigma \cdot \xi) = -|\xi|^2 = -1.$$

This proves (7.5).

$\square$

Proposition 7.4. *The operator A, equipped with the domain*

$$\mathcal{D}(A) = H_0^1(\Omega)^4 \cap H^2(\Omega)^4,$$

is a positive self-adjoint operator on $L^2(\Omega)^4$.

Proof. We shall consider the Friedrichs extension of A on $C_0^\infty(\Omega)^4$, denoted also by A, which has the domain

$$\mathcal{D}(A) = \mathcal{D}(Q) \cap \mathcal{D}(A_{\max}).$$

Here

$$Q(\varphi, \varphi) = \langle A\varphi, \varphi \rangle_{L^2(\Omega)^4}$$

is the quadratic form associated with the operator A. The domain $\mathcal{D}(Q)$, also the form domain of A, is the completion of $C_0^\infty(\Omega)^4$ with respect to the norm $\|\varphi\| = \sqrt{Q(\varphi, \varphi)}$. The maximal realization $A_{\max}$ of the operator A is defined by

$$\mathcal{D}(A_{\max}) = \{u \in L^2(\Omega)^4 : Au \in L^2(\Omega)^4\}.$$

Let us now show that

$$\mathcal{D}(Q) = H_0^1(\Omega)^4. \tag{7.6}$$

Indeed, it is easy to see that the norm $\|\cdot\|$ is equivalent to the following norm

$$\|\mathcal{L}_0 \cdot\|_{L^2(\Omega)^4} + \|\cdot\|_{L^2(\Omega)^4}. \tag{7.7}$$

Since $\|\mathcal{L}_0 \cdot\|_{L^2(\Omega)^4} \leq C\|\nabla \cdot\|_{L^2(\Omega)^2}$, we have

$$H_0^1(\Omega)^4 \subset \mathcal{D}(Q).$$

On the other hand, it follows from [22, Proposition 4.2] that the completion of $C^\infty(\overline{\Omega})^4$ with respect to the norm (7.7) is the space

$$\mathcal{H}(\Omega) = \{u \in L^2(\Omega)^4 : \mathcal{L}_0 u \in L^2(\Omega)^4\}.$$

Thus,

$$\mathcal{D}(Q) \subset \mathcal{H}(\Omega).$$

It is shown in [22, Proposition 4.6] that the trace map

$$\tau : C^\infty(\overline{\Omega})^4 \to C^\infty(\partial\Omega)^4, \quad u \mapsto u|_{\partial\Omega},$$

extends uniquely to a bounded map on $\mathcal{H}(\Omega)$. It follows from [22, Proposition 4.6] that for any $u \in \mathcal{D}(Q)$, $\tau u = 0$. Hence, [22, Proposition 4.10] implies that $u \in H_0^1(\Omega)^4$. This proves (7.6).

Hence,

$$\mathcal{D}(A) = \{u \in H_0^1(\Omega)^4 : Au \in L^2(\Omega)^4\}.$$

As the operator A is strongly elliptic and Ω has a smooth boundary, by elliptic regularity, see for instance [13, Section 7.5], $\mathcal{D}(A) = H_0^1(\Omega)^4 \cap H^2(\Omega)^4$.

$\square$

It follows from Proposition 7.4 that for any $\lambda \in \mathbb{R}$, the operator T_λ, equipped with the domain $\mathcal{D}(A)$ is a self-adjoint operator on $L^2(\Omega)^4$, and the form domain of T_λ is $H_0^1(\Omega)^4$.

Theorem 7.5. *The set of transmission eigenvalues for* (7.4) *is discrete.*

Proof. First note that the operator

$$-\lambda B + \lambda^2 C : \mathcal{D}(A) \to L^2(\Omega)$$

is compact. Hence, the operator $T_\lambda : \mathcal{D}(A) \to L^2(\Omega)$ is Fredholm of index 0, invertible at $\lambda = 0$. Thus, by analytic Fredholm theory,

$$T_\lambda^{-1} : L^2(\Omega) \to \mathcal{D}(A), \quad \lambda \in \mathbb{C},$$

is a meromorphic family of operators with residues of finite rank. This proves the claim.

$\square$

As before, we see that the multiplicity of a transmission eigenvalue $\lambda \in \mathbb{R}$ is finite.

Theorem 7.6. *Let V be a matrix-valued potential as above. Then there exists an infinite set of real transmission eigenvalues for* (7.4).

Proof. First notice that Proposition 6.2 continues to be valid for (7.4). It is therefore sufficient to prove the existence of transmission eigenvalues for the following problem,

$$
\begin{aligned}
(\mathcal{L}_0(D) - \lambda I_4)v &= 0 \quad \text{in} \quad B_\varepsilon(0), \\
(\mathcal{L}_0(D) - \lambda(1 + \delta)I_4)w &= 0 \quad \text{in} \quad B_\varepsilon(0), \\
v - w &\in H_0^1(B_\varepsilon(0))^4.
\end{aligned}
\tag{7.8}
$$

Here $\varepsilon > 0$, $\delta > 0$ and $B_\varepsilon(0) \subset \mathbb{R}^3$ is an open ball of radius ε centered at the origin.

When considering (7.8), we let $0 \neq \lambda \in \mathbb{R}$ and study

$$(\mathcal{L}_0(D) - \lambda I_4)v = 0,$$

where

$$v = \begin{pmatrix} v_+ \\ v_- \end{pmatrix}, \quad v_\pm = \begin{pmatrix} v_\pm^1 \\ v_\pm^2 \end{pmatrix}.$$

Then we have

$$
\begin{aligned}
-\lambda v_+ + \sigma \cdot D v_- &= 0, \\
\sigma \cdot D v_+ - \lambda v_- &= 0.
\end{aligned}
$$

As

$$\sigma \cdot D = \begin{pmatrix} D_3 & D_1 - iD_2 \\ D_1 + iD_2 & -D_3 \end{pmatrix},$$

we get

$$v_+^1 = \frac{1}{\lambda}(D_3 v_-^1 + (D_1 - iD_2)v_-^2)$$

$$v_+^2 = \frac{1}{\lambda}((D_1 + iD_2)v_-^1 - D_3 v_-^2),$$

$$\begin{pmatrix} -\Delta - \lambda^2 & 0 \\ 0 & -\Delta - \lambda^2 \end{pmatrix} \begin{pmatrix} v_-^1 \\ v_-^2 \end{pmatrix} = 0.$$

Considering the equation

$$(\mathcal{L}_0(D) - \lambda(1 + \delta)I_4)w = 0,$$

similarly, we obtain

$$w_+^1 = \frac{1}{\lambda(1 + \delta)}(D_3 w_-^1 + (D_1 - iD_2)w_-^2)$$

$$w_+^2 = \frac{1}{\lambda(1 + \delta)}((D_1 + iD_2)w_-^1 - D_3 w_-^2),$$

$$\begin{pmatrix} -\Delta - \lambda^2(1 + \delta)^2 & 0 \\ 0 & -\Delta - \lambda^2(1 + \delta)^2 \end{pmatrix} \begin{pmatrix} w_-^1 \\ w_-^2 \end{pmatrix} = 0.$$

Notice that to prove the existence of real transmission eigenvalues for the problem (7.8), it suffices to restrict our attention to solutions v, w of (7.8) such that $v_-^1 = v_-^2 = f(r)$ and $w_-^1 = w_-^2 = g(r)$, $r = |x|$, are spherically symmetric solutions of the following interior transmission problem,

$$(-\Delta - \lambda^2)f = 0 \quad \text{in} \quad B_\varepsilon(0),$$
$$(-\Delta - \lambda^2(1 + \delta)^2)g = 0 \quad \text{in} \quad B_\varepsilon(0),$$
$$f - g = 0 \quad \text{on} \quad \partial B_\varepsilon(0), \qquad (7.9)$$
$$\partial_r f = \frac{\partial_r g}{1 + \delta} \quad \text{on} \quad \partial B_\varepsilon(0).$$

It is clear then that for such solutions, the boundary conditions

$$v_+^1 = w_+^1 \quad \text{on} \quad \partial B_\varepsilon(0),$$
$$v_+^2 = w_+^2 \quad \text{on} \quad \partial B_\varepsilon(0).$$

are satisfied.

Since

$$(\Delta + \lambda^2)f(r) = f''(r) + \frac{2}{r}f'(r) + \lambda^2 f(r) = 0,$$

f must be of the form

$$f(x) = c_0 j_0(\lambda r),$$

where j_0 is the spherical Bessel function of order zero and c_0 is a constant. In the same way,

$$g(x) = c_1 j_0(\lambda(1 + \delta)r),$$

The boundary conditions in (7.9) require that

$$c_0 j_0(\lambda \varepsilon) = c_1 j_0(\lambda(1+\delta)\varepsilon),$$
$$c_0 j_0'(\lambda \varepsilon) = c_1 j_0'(\lambda(1+\delta)\varepsilon).$$

A nontrivial solution of this system exists if and only if

$$\det \begin{pmatrix} j_0(\lambda \varepsilon) & -j_0(\lambda(1+\delta)\varepsilon) \\ j_0'(\lambda \varepsilon) & -j_0'(\lambda(1+\delta)\varepsilon) \end{pmatrix} = 0. \tag{7.10}$$

Since

$$j_0(r) = \frac{\sin r}{r}, \quad j_0'(r) = \frac{\cos r}{r} + \mathcal{O}(1/r^2),$$

(7.10) implies that

$$\sin(\lambda \delta \varepsilon) + \mathcal{O}(1/\lambda) = 0, \quad \lambda \to \infty. \tag{7.11}$$

The existence of an infinite set of values λ such that (7.11) holds is clear as $\sin(\lambda \delta \varepsilon)$ is a periodic function taking positive and negative values. Each such λ is a transmission eigenvalue for (7.8) and this completes the proof.

□

8. Generalized Rellich theorem

In the last two sections, which do not depend on the material in Section 6 and 7, we would like to explain the connection between interior transmission eigenvalues and scattering theory. It is going to be provided by a generalization of the classical Rellich theorem, proved in [17].

Let us start by summarizing the basic features of general scattering theory following [15, Chapter 14]. Let P_0 be a partial differential operator in $\mathbb{R}^n$ of order $m \geq 2$ with constant real coefficients,

$$P_0(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha, \quad a_\alpha \in \mathbb{R}, \quad D_j = -i \frac{\partial}{\partial x_j}, \quad j = 1, \ldots, n.$$

Assume that $\Lambda(P_0) = \{0\}$, see (2.4), and that $P_0 = P_0(D)$ is simply characteristic, i.e.

$$\sum_{|\alpha| \leq m} |P_0^{(\alpha)}(\xi)| \leq C(\sum_{|\alpha| \leq 1} |P_0^{(\alpha)}(\xi)| + 1), \quad C > 0.$$

Examples are hypoelliptic operators [15, Chapter 11] and operators of real principal type [14, Chapter 8].

In order to describe mapping properties of the boundary values of the L^2-resolvent of P_0, we follow [15, Chapter 14] and introduce the following Banach spaces.

Define

$$B = \{v \in L^2(\mathbb{R}^n) : \|v\|_B = \sum_{j=1}^{\infty} R_j^{1/2} (\int_{\Omega_j} |v|^2 dx)^{1/2} < \infty\},$$

$$B^* = \{u \in L^2_{\text{loc}}(\mathbb{R}^n) : \|u\|_{B^*} = \sup_{j>0} R_j^{-1/2} (\int_{\Omega_j} |u|^2 dx)^{1/2} < \infty\},$$

where

$$R_0 = 0, \quad R_j = 2^{j-1}, \quad j = 1, 2, \ldots,$$
$$\Omega_j = \{x \in \mathbb{R}^n : R_{j-1} < |x| < R_j\}, \quad j = 1, 2, \ldots.$$

The space B^* is the dual of B and we have

$$L^2_\delta \subset B \subset L^2 \subset B^* \subset L^2_{-\delta}, \quad \delta > 1/2.$$

The space of C_0^∞-functions is dense in B but not in B^*. Its closure in B^* is denoted by $\overset{\circ}{B}{}^*$. Then $u \in L^2_{\text{loc}}(\mathbb{R}^n)$ belongs to $\overset{\circ}{B}{}^*$ if and only if

$$\int_{|x|<R} |u|^2 dx/R \to 0, \quad R \to \infty.$$

We also define the Sobolev space version of B^*, associated to $P_0(D)$,

$$B^*_{P_0} = \{u \in B^* : P_0^{(\alpha)}(D)u \in B^*, \forall \alpha\}.$$

Let $Z(P_0)$ be the (necessarily finite) set of critical values of P_0, i.e.

$$Z(P_0) = \{\lambda : \exists \xi \in \mathbb{R}^n \text{ s.t. } \nabla P_0(\xi) = 0, P_0(\xi) = \lambda\}.$$

For $z \in \overline{\mathbb{C}^\pm} \setminus Z(P_0)$, the resolvent $R_0(z) = (P_0 - zI)^{-1}$ of the simply characteristic operator P_0 extends to a continuous map

$$R_0(z) : B \to B^*_{P_0}.$$

Here $\overline{\mathbb{C}^\pm} = \{z : \pm\text{Im } z \geq 0\}$. For $\lambda \in \mathbb{R} \setminus Z(P_0)$, the boundary values of the resolvent are given by

$$R_0(\lambda \pm i0)f = \lim_{\varepsilon \to 0^+} F^{-1}((P_0 - \lambda \mp \varepsilon i)^{-1}F(f)),$$

where F stands for the Fourier transformation.

We say that $u \in B^*$ is outgoing (incoming) if $u = R(\lambda + i0)f$ ($u = R(\lambda - i0)f$), $f \in B$ and $\lambda \in \mathbb{R} \setminus Z(P_0)$. If u is outgoing or incoming then $(P_0 - \lambda)u = f$.

For $\lambda \notin Z(P_0)$, the level set $M_\lambda = \{\xi \in \mathbb{R}^n : P_0(\xi) = \lambda\}$ is an $(n-1)$-dimensional C^∞ submanifold of $\mathbb{R}^n$.

It is known that u is both outgoing and incoming, i.e. $u = R(\lambda + i0)f = R(\lambda - i0)f$, if and only if $F(f) = \hat{f} \equiv 0$ on M_λ.

We shall consider multiplicative perturbations of P_0 given by $V \in L^\infty(\mathbb{R}^n)$ with compact support. Such perturbations satisfy the short range condition introduced in [15, Section 14.4], i.e.

$$V : B^*_{P_0} \to B$$

is compact.

In order to define the scattering amplitude, we recall the following fundamental result [15, Theorem 14.6.8].

Theorem 8.1. *Assume that* $\lambda \in \mathbb{R} \setminus Z(P_0)$. *If* $u \in B^*_{P_0}$ *satisfies*

$$(P_0 + V - \lambda)u = 0,$$

then

$$u = u_\pm - R_0(\lambda \mp i0)Vu \qquad\qquad (8.1)$$

where

$$\widehat{u}_\pm = v_\pm \delta(P_0 - \lambda) = v_\pm \frac{dS}{|P_0'|} \quad and \quad v_\pm \in L^2(M_\lambda, dS).$$

The map $v_- \mapsto v_+$ *is a continuos bijection which extends to a unitary map*

$$\Sigma_\lambda : L^2(M_\lambda, \frac{dS}{|P_0'|}) \to L^2(M_\lambda, \frac{dS}{|P_0'|}), \quad \Sigma_\lambda(v_-) = v_+.$$

We call v_- the incoming wave and v_+ the outgoing wave. The unitary map Σ_λ is the scattering matrix for the energy λ and

$$A_\lambda = I - \Sigma_\lambda$$

is the scattering amplitude.

The following result is well-known and its proof is included for completeness only.

Lemma 8.2. *Assume that* $\lambda \in \mathbb{R} \setminus Z(P_0)$ *and* $u \in B^*_{P_0}$ *satisfies the equation* $(P_0 + V - \lambda)u = 0$. *Then the scattering amplitude* A_λ *can be expressed through the Fourier transform of* Vu *as follows,*

$$A_\lambda v_-(\xi) = 2\pi i \widehat{V u}(\xi), \quad \xi \in M_\lambda.$$

Proof. By [15, Theorem 14.6.8], we get the following Lippmann-Schwinger equation for u,

$$u = u_\pm - R_0(\lambda \mp i0)Vu$$

where

$$\widehat{u}_\pm = v_\pm \delta(P_0 - \lambda) = v_\pm \frac{dS}{|P_0'|} \quad and \quad v_\pm \in L^2(M_\lambda, dS).$$

Thus,

$$u_+ - u_- = (R_0(\lambda - i0) - R_0(\lambda + i0))Vu.$$

Applying the Fourier transform, we have

$$(v_+ - v_-)\frac{dS}{|P_0'|} = \Big(\frac{1}{P_0(\xi) - \lambda + i0} - \frac{1}{P_0(\xi) - \lambda - i0}\Big)\widehat{Vu}$$

$$= 2\pi i\delta(P_0(\xi) - \lambda)\widehat{Vu} = 2\pi i\widehat{Vu}\frac{dS}{|P_0'|}$$

that proves the claim.   □

In order to describe the connection between transmission eigenvalues and the scattering amplitude, we shall now review the generalized Rellich theorem. Recall that a classical theorem of Rellich states that if v satisfies $(\Delta + k^2)v = 0$ for $|x| > R_0$ and $v(x)|x|^{(n-1)/2} \to 0$ as $x \to \infty$, then $v(x) = 0$ for $|x| \geq R_0$.

A far-reaching generalization of this result to broad classes of differential operators with constant coefficients has been given in [17]. Let us now state a much simplified version of [17, Corollary 3.2]. See also [1].

Theorem 8.3. *Assume that λ is not a critical value of $P_0(\xi)$, $\xi \in \mathbb{R}^n$, and that there is a factorization*

$$P_0(\zeta) - \lambda = cP_1^{m_1}(\zeta) \cdots P_k^{m_k}(\zeta), \quad c \in \mathbb{R},$$

for which every factor $P_i(\zeta)$ has real coefficients and is algebraically irreducible over $\mathbb{C}^n$. Assume furthermore that each $P_i(\zeta)$ has a non-empty set of real zeros. If $u \in \mathcal{S}' \cap L_{loc}^2$ is a solution of

$$(P_0(D) - \lambda)u = f, \tag{8.2}$$

with $f \in L_{comp}^2$ and $u \in \overset{\circ}{B}^*$, then u has a compact support and

$$chsupp\ (u) = chsupp\ (f),$$

where ch stands for the convex hull.

Remark 8.1. If $P_0(\zeta) - \lambda$ has an irreducible factor which has no simple real zero, then it was furthermore proved in [17] that for any integer N one can find $u \in L^\infty \cap C^\infty$ satisfying (8.2) with f being compactly supported and $u(x) = o(|x|^{-N})$ but u not compactly supported.

To illustrate the main ideas involved in the proof of Theorem 8.3, for the convenience of the reader, we shall include a proof of the special case when $k = 1$, $m = 1$ and $c = 1$. In doing so we shall follow [17] closely.

Proof. Set $P_\lambda(\zeta) = P_0(\zeta) - \lambda$, $\zeta \in \mathbb{C}^n$. The set of real zeros of the polynomial $P_\lambda(\xi)$ is equal to $M_\lambda = \{\xi \in \mathbb{R}^n : P_\lambda(\xi) = 0\}$. By the hypothesis of the theorem, $M_\lambda \neq \emptyset$ is an $(n-1)$-dimensional submanifold of $\mathbb{R}^n$.

Let $\xi_0 \in M_\lambda$. Assume, as we may, that $\partial_{\xi_n} P_0(\xi_0) \neq 0$, and write $\xi = (\xi', \xi_n) \in \mathbb{R}^n$, $\xi' \in \mathbb{R}^{n-1}$. By the implicit function theorem, there is an analytic function $g : \mathbb{R}^{n-1} \to \mathbb{R}$, $g(\xi') = \sum a_\alpha(\xi' - \xi_0')^\alpha$, defined locally near ξ_0', such that

$$M_\lambda = \{(\xi', g(\xi')) : \xi' \in \mathbb{R}^{n-1}\},$$

locally near $\xi_0 \in M_\lambda$. Hence, the series $g(\zeta') = \sum a_\alpha(\zeta' - \xi_0')^\alpha$ also converges for $\zeta' \in \mathbb{C}^{n-1}$ near ξ_0'.

Let $M_\lambda^{\mathbb{C}}$ be the zero set of P_λ in $\mathbb{C}^n$, i.e.

$$M_\lambda^{\mathbb{C}} = \{\zeta \in \mathbb{C}^n : P_\lambda(\zeta) = 0\}.$$

Then let us show that locally near ξ_0, we have

$$M_\lambda^{\mathbb{C}} = \{(\zeta', g(\zeta')) : \zeta' \in \mathbb{C}^{n-1}\}.$$

Indeed, $P_\lambda(\xi', g(\xi')) = 0$ for any $\xi' \in \mathbb{R}^{n-1}$ near ξ_0' and, hence, since P_λ and g are analytic, we get that $P_\lambda(\zeta', g(\zeta')) = 0$ for any $\zeta' \in \mathbb{C}^{n-1}$ near ξ_0'.

Now since $u \in \overset{\circ}{B}{}^*$, the Fourier transform $\widehat{f}$ vanishes on M_λ (cf. [15, Theorem 14.3.6]). Thus, by the analyticity of $\widehat{f}$ and g, we have that $\widehat{f}$ vanishes identically on an open neighborhood of ξ_0 in $M_\lambda^{\mathbb{C}}$.

Denote by

$$A = \{\zeta \in \mathbb{C}^n : \widehat{f}(\zeta) = 0\}$$

the set of the complex zeros of $\widehat{f}$. Notice that the sets A and $M_\lambda^{\mathbb{C}}$ are analytic. As we have assumed that $P_\lambda(\zeta)$ is algebraically irreducible over $\mathbb{C}^n$, the set $M_\lambda^{\mathbb{C}}$ is algebraically irreducible, and hence it is analytically irreducible as well, [26]. Moreover, $A \cap M_\lambda^{\mathbb{C}}$ contains an open neighborhood of ξ_0 in $M_\lambda^{\mathbb{C}}$. Then it follows from [7, Corollary 2, Section 5.3] that

$$M_\lambda^{\mathbb{C}} \subset A. \tag{8.3}$$

Thus, the function

$$\frac{\widehat{f}(\zeta)}{P_\lambda(\zeta)}$$

is entire in $\mathbb{C}^n$. An application of [14, Theorem 7.3.2] shows that there exists $v \in L^2_{comp}$ such that

$$P_\lambda(D)v = f.$$

Hence,

$$P_\lambda(D)(u - v) = 0,$$

and since $u \in \overset{\circ}{B}{}^*$, [14, Theorem 7.1.27] implies that $u = v$. $\qquad\qquad\square$

9. Injectivity of the scattering amplitude and transmission eigenvalues

In this section we shall assume that the operator P_0 is hypoelliptic and $\lambda \in \mathbb{R}$ is such that $P_0 - \lambda$ satisfies the assumptions of Theorem 8.3. Let $V \in L^\infty(\mathbb{R}^n)$ be real-valued compactly supported in $\mathbb{R}^n$ with supp $(V) = \overline{\Omega}$, where $\Omega \subset \mathbb{R}^n$ is a bounded convex domain which is of class C^∞, and $V \geq \delta > 0$ a.e. in Ω.

Consider the interior transmission problem,

$$
\begin{aligned}
(P_0 - \lambda)v &= 0 \quad \text{in} \quad \Omega, \\
(P_0 + V - \lambda)w &= 0 \quad \text{in} \quad \Omega, \\
v - w &\in H_0^{P_0}(\Omega).
\end{aligned}
\tag{9.1}
$$

Theorem 9.1. *There is a solution (v, w) of (9.1) with $0 \neq v \in B^*$ if and only if the scattering amplitude A_λ is not injective.*

Proof. Assume that A_λ is not injective. Then there exists a solution $u \in B_{P_0}^*$ of $(P_0 + V - \lambda)u = 0$ in $\mathbb{R}^n$ such that $u = u_- - u_s$ with $u_- \neq 0$ and the scattered wave

$$
u_s := R_0(\lambda + i0)Vu
$$

being both incoming and outgoing. By the mapping properties of the resolvent $R_0(\lambda + i0) : B \to B_{P_0}^*$, we see that $u_- \in B_{P_0}^*$. Now since the scattered wave u_s is both incoming and outgoing by Theorem [15, Theorem 14.3.6] $u_s \in \overset{\circ}{B}{}^*$. Moreover, $(P_0 - \lambda)u_s = Vu$. Now Theorem 8.3 implies that u_s has compact support and supp $(u_s) \subset \overline{\Omega}$, thanks to the convexity of Ω. Since $u, u_s \in L_{\text{loc}}^2(\mathbb{R}^n)$ and P_0 is hypoelliptic, by hypoelliptic regularity [15, Theorem 11.1.8], we have that $u_s \in B_{2,\widetilde{P_0}}^{loc}(\mathbb{R}^n)$. As supp $(u_s) \subset \overline{\Omega}$, a regularization argument shows that u_s can be approximated by a sequence of $C_0^\infty(\Omega)$-functions, so that $u_s|_\Omega \in H_0^{P_0}(\Omega)$. Now setting $v = u_-|_\Omega \neq 0$ and $w = u|_\Omega$, we get a nontrivial solution to (9.1) .

Assume conversely that the problem (9.1) admits a non-trivial solution (v, w) with $0 \neq v \in B^*$. Then by [15, Theorem 14.3.3], we have $\widehat{v} = v_- dS$ with $v_- \in L^2(M_\lambda, dS)$. Since P_0 is hypoelliptic, the surface M_λ is compact, and thus, [15, Theorem 14.3.8] implies that $v \in B_{P_0}^*$. As $v - w \in H_0^{P_0}(\Omega)$, we get that $w \in B_{P_0}^*$ and $v - w \in \overset{\circ}{B}{}^*$. Now $(P_0 - \lambda)(v - w) = Vw$ and [15, Theorem 14.3.6] yields that the Fourier transform $\widehat{Vw} = 0$ on M_λ. Hence, applying Lemma 8.2, we get that $A_\lambda v_- = 0$. $\square$

Remark 9.1. The convexity assumption on Ω can be removed if we require that P_0 is elliptic and $\mathbb{R}^n \setminus \overline{\Omega}$ is connected.

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